IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Hamiltonian formulation of the inverse scattering method of the modified KdV equation under the non-vanishing boundary condition u(x, t) to b as x to +or- infinity

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1988 J. Phys. A: Math. Gen. 21 3575

(http://iopscience.iop.org/0305-4470/21/18/013)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 01/06/2010 at 06:00

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 21 (1988) 3575-3592. Printed in the UK

Hamiltonian formulation of the inverse scattering method of the modified KdV equation under the non-vanishing boundary condition $u(x, t) \to b$ as $x \to \pm\infty$

T C Au Yeung† and P C W Fung‡

† Department of Physics, University of Hong Kong, Hong Kong and Department of Mathematical Studies, Hong Kong Polytechnic, Hong Kong
‡ Department of Physics, University of Hong Kong, Hong Kong

Received 25 January 1988, in final form 5 May 1988

Abstract. Previously it was shown that the inverse scattering transform for the modified KdV equation is a canonical transformation under the vanishing boundary condition, with the scattering data being essentially a set of action-angle variables (Flaschka and Newell). Recently, we have developed the inverse scattering method for the modified KdV equation under the non-vanishing condition $u(x, t) \to b$, as $x \to \pm\infty$, where b is an arbitrary constant. In this investigation we prove that, under the above non-vanishing condition, the inverse scattering transform for the modified KdV equation is a canonical transformation and the scattering data are essentially a set of action-angle variables. Hence the modified KdV equation is completely integrable under the non-vanishing condition $u(x, t) \to b$ as $x \to \pm\infty$.

1. Introduction

The inverse scattering method of the MKdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{1}$$

was established under the non-vanishing condition

$$u(x, t) \to b \qquad \text{as} \qquad x \to \pm\infty \tag{2}$$

where b is an arbitrary non-zero constant (Au Yeung *et al* 1984). The time variations of the scattering data were found to satisfy an infinite system of ordinary differential equations which can be trivially solved (Au Yeung *et al* 1984). This property is similar to that of the vanishing case $u(x, t) \to 0$ as $x \to \pm\infty$ (Wadati 1972). Now, in the vanishing case the inverse scattering transform for the MKdV equation is a canonical transformation, with the scattering data being essentially a set of action-angle variables (Flaschka and Newell 1975, Newell 1980). It is then suggestive to ask whether the inverse scattering transform under the non-vanishing condition $u(x, t) \to b$ as $x \to \pm\infty$ is a canonical transformation and whether the scattering data are essentially a set of action-angle variables.

In this paper we look at the above questions. We shall show that, under the non-vanishing condition (2), the MKdV equation represents a Hamiltonian system and the inverse scattering transform is a canonical transformation, with the scattering data being essentially a set of action-angle variables. We also obtain an infinite set of conserved integrals of the MKdV equation under the non-vanishing condition (2).

0305-4470/88/183575+18$02.50 © 1988 IOP Publishing Ltd

For the convenience of discussion, we give in § 2 a brief review of some aspects of the inverse scattering transform of the MKdV equation under the non-vanishing condition (2). In § 3 the MKdV equation is shown to be a Hamiltonian under the non-vanishing condition. We prove in § 4 that the inverse scattering transform of the MKdV equation under the non-vanishing condition is a canonical transformation, with the scattering data being of the action–angle type. In § 4 we also obtain an infinite set of conserved integrals of the MKdV equation under the non-vanishing condition. Some conclusions will be given in § 5.

2. The inverse scattering transform of the MKdV equation under the non-vanishing condition $u(x, t) \to b$ as $x \to \pm\infty$

The inverse scattering method of the MKdV equation under the non-vanishing condition (2) associates with the AKNS problem

$$\frac{\partial V}{\partial x} = \begin{pmatrix} -\mathrm{i}\lambda & q(x) \\ r(x) & \mathrm{i}\lambda \end{pmatrix} V \qquad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \tag{3}$$

where

$$r(x) = -q(x) \qquad q(x) = u(x) \tag{4}$$

(Kawata and Inoue 1977, 1978, Au Yeung *et al* 1984). In the method solutions $\Phi^{\pm}(\lambda, \xi) = (\phi_1^{\pm}(\lambda, \xi), \phi_2^{\pm}(\lambda, \xi))$ to the AKNS problem are defined by the following boundary conditions:

$$\Phi^{\pm}(\lambda, \xi) \to \begin{pmatrix} -\mathrm{i}q^{\pm} & \lambda - \xi \\ \lambda - \xi & \mathrm{i}r^{\pm} \end{pmatrix} \begin{pmatrix} \exp(-\mathrm{i}\xi x) & 0 \\ 0 & \exp(\mathrm{i}\xi x) \end{pmatrix} \qquad \text{as} \qquad x \to \pm\infty \tag{5}$$

where

$$q^{\pm} = -r^{\pm} = b \tag{6}$$

and

$$\xi = (\lambda^2 - \lambda_0^2)^{1/2} \qquad \lambda_0^2 = -b^2. \tag{7}$$

The scattering matrix

$$S(\lambda, \xi) = \begin{pmatrix} S_{11}(\lambda, \xi) & S_{12}(\lambda, \xi) \\ S_{21}(\lambda, \xi) & S_{22}(\lambda, \xi) \end{pmatrix}$$

is defined by

$$\Phi^{-}(\lambda, \xi) = \Phi^{+}(\lambda, \xi) \cdot S(\lambda, \xi). \tag{8}$$

The function $\xi = (\lambda^2 - \lambda_0^2)^{1/2}$ is multivalued and it is set to be single-valued by introducing two Riemann surfaces. A cut is set in the region $(-\lambda_0, \lambda_0)$ of the pure imaginary axis. The upper (lower) Riemann surface is defined to be $\xi \to \lambda$ ($\xi \to -\lambda$) as $|\lambda| \to \infty$ and the sign of Im ξ is equal (opposite) to the sign of Im λ on the upper (lower) surface.

The scattering element $S_{11}(\lambda, \xi)$ is analytic in λ in the region Im $\xi > 0$ of the two Riemann surfaces (Kawata and Inoue 1977, 1978) and the zeros of $S_{11}(\lambda, \xi)$ in this region are just the eigenvalues of the AKNS problem. Let λ_k, $k = 1, 2, \ldots, N$, be the zeros of $S_{11}(\lambda, \xi)$ in the region Im $\xi > 0$ of the upper Riemann surface.

Define $m(\lambda)$ and $\rho_1(\lambda, \xi)$ by

$$m(\lambda) = S_{21}(\lambda, \xi)/\xi \cdot \mathrm{d}S_{11}/\mathrm{d}\lambda \tag{9}$$

and

$$\rho_1(\lambda, \xi) = S_{21}(\lambda, \xi)/S_{11}(\lambda, \xi). \tag{10}$$

The set $\{\lambda_k, m(\lambda_k),\ k = 1, 2, \ldots, N;\ \rho_1(\lambda, \xi)\}$ is the scattering data under the non-vanishing condition (2) (Au Yeung *et al* 1984). The mapping $u(x) \to \{\lambda_k, m(\lambda_k), \rho_1(\lambda, \xi)\}$ is the inverse scattering transform of the MKdV equation under the non-vanishing condition.

Certain symmetries exist in the scattering matrix $S(\lambda, \xi)$ (Au Yeung *et al* 1984). These symmetries are as below:

$$\begin{aligned} S_{11}(\lambda, \xi) &= \frac{q^-}{q^+} S_{22}(\lambda, -\xi) \qquad & S_{12}(\lambda, \xi) &= -\frac{r^-}{q^+} S_{21}(\lambda, -\xi) \\ S_{21}(\lambda, \xi) &= -\frac{q^+}{r^-} S_{12}(\lambda, -\xi) \qquad & S_{22}(\lambda, \xi) &= \frac{q^+}{q^-} S_{11}(\lambda, -\xi) \end{aligned} \tag{11}$$

and

$$\begin{aligned} S_{11}(\lambda, \xi) &= S_{22}(-\lambda, -\xi) \qquad & S_{12}(\lambda, \xi) &= -S_{21}(-\lambda, -\xi) \\ S_{21}(\lambda, \xi) &= -S_{12}(-\lambda, -\xi) \qquad & S_{22}(\lambda, \xi) &= S_{11}(-\lambda, -\xi) \end{aligned} \tag{12}$$

and also

$$\begin{aligned} S_{11}(\lambda, \xi) &= S_{22}(\lambda^*, \xi^*)^* \qquad & S_{12}(\lambda, \xi) &= -S_{21}(\lambda^*, \xi^*)^* \\ S_{21}(\lambda, \xi) &= -S_{12}(\lambda^*, \xi^*)^* \qquad & S_{22}(\lambda, \xi) &= S_{11}(\lambda^*, \xi^*)^*. \end{aligned} \tag{13}$$

Symmetries (11), (12) and (13) imply the following relations:

$$\frac{q^-}{q^+} S_{11}(\lambda, \xi) = S_{11}(-\lambda, \xi) \qquad S_{21}(\lambda, \xi) = -\frac{q^+}{q^-} S_{21}(-\lambda, \xi) \tag{14}$$

$$m(-\lambda) = m(\lambda) \tag{15}$$

$$S_{11}(\lambda, \xi) = S_{11}(-\lambda^*, -\xi^*)^* \qquad S_{21}(\lambda, \xi) = S_{21}(-\lambda^*, -\xi^*)^* \tag{16}$$

$$m(\lambda) = m(-\lambda^*)^* \tag{17}$$

$$|S_{11}(\lambda, \xi)|^2 + |S_{21}(\lambda, \xi)|^2 = 1 \tag{18a}$$

for real positive λ and

$$|S_{11}(\lambda, \xi)|^2 - |S_{21}(\lambda, \xi)|^2 = 1 \tag{18b}$$

for pure imaginary λ with $0 < -\mathrm{i}\lambda < |b|$.

3. The MKdV equation as a Hamiltonian under the non-vanishing condition $u(x, t) \to b$ as $x \to \pm\infty$

We shall show that the MKdV equation represents a Hamiltonian system under the non-vanishing condition (2). Consider the functional $H[u]$ defined by

$$H[u] = -\tfrac{1}{2} \int_{-\infty}^{\infty} [u^4 - u_x^2 - b^4]\, \mathrm{d}x. \tag{19}$$

This functional is well defined under the non-vanishing condition $u(x, t) \to b$ as $x \to \pm\infty$. The variational derivative $\delta H/\delta u(x)$ is, from (19),

$$\delta H/\delta u(x) = -2u^3 - u_{xx}. \tag{20}$$

From (20) we see that the equation

$$u_t = \frac{\partial}{\partial x}\frac{\delta H}{\delta u(x)} \tag{21}$$

is identical to the MKdV equation. In view of the well known property of the operator $\partial/\partial x$ (Gardner 1971) we conclude that equation (21) is a Hamiltonian. Hence the MKdV equation represents a Hamiltonian system under the non-vanishing condition, and the Hamiltonian is equal to $H[u]$. The phase space consists of functions $u(x)$ which tend to b at infinity. Also, the corresponding Poisson bracket of any two functionals $f[u]$ and $g[u]$ is given by

$$\{f, g\} = \int_{-\infty}^{\infty} \frac{\delta f}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta g}{\delta u(x)}\, dx. \tag{22}$$

4. The inverse scattering transform of the MKdV equation as a canonical transformation under the non-vanishing condition $u(x, t) \to b$ as $x \to \pm\infty$

In this section we will show that under the non-vanishing condition (2), the inverse scattering transform for the MKdV equation is a canonical transformation. As already discussed in § 2 the scattering matrix

$$S(\lambda, \xi) = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

is defined by

$$\begin{aligned} \phi_1^-(\lambda, \xi) &= S_{11}(\lambda, \xi) \cdot \phi_1^+(\lambda, \xi) + S_{21}(\lambda, \xi) \cdot \phi_2^+(\lambda, \xi) \\ \phi_2^-(\lambda, \xi) &= S_{12}(\lambda, \xi)\phi_1^+(\lambda, \xi) + S_{22}(\lambda, \xi)\phi_2^+(\lambda, \xi) \end{aligned} \tag{8}$$

where $\phi_1^\pm(\lambda, \xi)$, $\phi_2^\pm(\lambda, \xi)$ are solutions to the AKNS problem (3) satisfying the following boundary conditions:

$$\begin{aligned} \phi_1^\pm(\lambda, \xi) &\to \begin{pmatrix} -\mathrm{i}q^\pm \\ \lambda - \xi \end{pmatrix} \exp(-\mathrm{i}\xi x) \qquad \text{as} \qquad x \to \pm\infty \\ \phi_2^\pm(\lambda, \xi) &\to \begin{pmatrix} \lambda - \xi \\ \mathrm{i}r^\pm \end{pmatrix} \exp(\mathrm{i}\xi x) \qquad \text{as} \qquad x \to \pm\infty \end{aligned} \tag{5a}$$

where

$$q^\pm = -r^\pm = b.$$

From (8) we have

$$\begin{aligned} S_{11}(\lambda, \xi) &= \frac{1}{2\xi(\lambda - \xi)} W(\phi_1^-(\lambda, \xi), \phi_2^+(\lambda, \xi)) \\ S_{21}(\lambda, \xi) &= \frac{-1}{2\xi(\lambda - \xi)} W(\phi_1^-(\lambda, \xi), \phi_1^+(\lambda, \xi)) \end{aligned} \tag{23}$$

where $W(f, g)$ for any two solutions

$$f=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad \text{and} \qquad g=\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

to the AKNS problem (3) is the Wronskian of f and g defined by

$$W(f, g) \equiv f_1 g_2 - f_2 g_1. \tag{24}$$

Now, using the symmetry (12) of the scattering matrix, we have from (8)

$$\begin{aligned} \phi_1^-(\lambda, \xi) &= S_{11}(\lambda, \xi)\phi_1^+(\lambda, \xi) + S_{21}(\lambda, \xi)\phi_2^+(\lambda, \xi) \\ \phi_2^-(\lambda, \xi) &= -S_{21}(-\lambda, -\xi)\phi_1^+(\lambda, \xi) + S_{11}(-\lambda, -\xi)\phi_2^+(\lambda, \xi). \end{aligned} \tag{25}$$

From (25) we obtain

$$\begin{aligned} \phi_1^+(\lambda, \xi) &= S_{11}(-\lambda, -\xi)\phi_1^-(\lambda, \xi) - S_{21}(\lambda, \xi)\phi_2^-(\lambda, \xi) \\ \phi_2^+(\lambda, \xi) &= S_{21}(-\lambda, -\xi)\phi_1^-(\lambda, \xi) + S_{11}(\lambda, \xi)\phi_2^-(\lambda, \xi). \end{aligned} \tag{26}$$

Using (25) and (26) we arrive at

$$\begin{aligned} \phi_1^+(\lambda, \xi) &\to S_{11}(-\lambda, -\xi)\begin{pmatrix} -\mathrm{i}q^- \\ \lambda-\xi \end{pmatrix}\exp(-\mathrm{i}\xi x) \\ &\quad - S_{21}(\lambda, \xi)\cdot\begin{pmatrix} \lambda-\xi \\ \mathrm{i}r^- \end{pmatrix}\exp(\mathrm{i}\xi x) \qquad \text{as} \qquad x\to-\infty \\ \phi_2^+(\lambda, \xi) &\to S_{21}(-\lambda, -\xi)\begin{pmatrix} -\mathrm{i}q^- \\ \lambda-\xi \end{pmatrix}\exp(-\mathrm{i}\xi x) \\ &\quad + S_{11}(\lambda, \xi)\cdot\begin{pmatrix} \lambda-\xi \\ \mathrm{i}r^- \end{pmatrix}\exp(\mathrm{i}\xi x) \qquad \text{as} \qquad x\to-\infty \\ \phi_1^-(\lambda, \xi) &\to S_{11}(\lambda, \xi)\begin{pmatrix} -\mathrm{i}q^+ \\ \lambda-\xi \end{pmatrix}\exp(-\mathrm{i}\xi x) \\ &\quad + S_{21}(\lambda, \xi)\cdot\begin{pmatrix} \lambda-\xi \\ \mathrm{i}r^+ \end{pmatrix}\exp(\mathrm{i}\xi x) \qquad \text{as} \qquad x\to\infty \\ \phi_2^-(\lambda, \xi) &\to -S_{21}(-\lambda, -\xi)\begin{pmatrix} -\mathrm{i}q^+ \\ \lambda-\xi \end{pmatrix}\exp(-\mathrm{i}\xi x) \\ &\quad + S_{11}(-\lambda, -\xi)\cdot\begin{pmatrix} \lambda-\xi \\ \mathrm{i}r^+ \end{pmatrix}\exp(\mathrm{i}\xi x) \qquad \text{as} \qquad x\to\infty. \end{aligned} \tag{27}$$

Next, using the AKNS problem (3) and the boundary conditions (5*a*) we obtain

$$\frac{\delta S_{11}(\lambda, \xi)}{\delta u(x)} = \frac{1}{2\xi(\lambda-\xi)}\,\phi_1^-(\lambda, \xi, x)^{\mathrm{T}}\cdot\phi_2^+(\lambda, \xi, x) \tag{28}$$

$$\frac{\delta S_{21}(\lambda, \xi)}{\delta u(x)} = \frac{-1}{2\xi(\lambda-\xi)}\,\phi_1^-(\lambda, \xi, x)^{\mathrm{T}}\cdot\phi_1^+(\lambda, \xi, x) \tag{29}$$

$$\frac{\delta\lambda_i}{\delta u(x)} = \frac{-1}{2\xi_i(\lambda_i-\xi_i)S'_{11}(\lambda_i)}\cdot\phi_1^-(\lambda_i, \xi_i, x)^{\mathrm{T}}\cdot\phi_2^+(\lambda_i, \xi_i, x) \qquad i=1, 2, \ldots, N \tag{30}$$

where λ_i, $i=1,2,\ldots,N$, are the zeros of $S_{11}(\lambda,\xi)$ in the region $\operatorname{Im}\xi>0$ of the upper Riemann surface and

$$S'_{11}(\lambda)\equiv\frac{\mathrm{d}S_{11}(\lambda,\xi)}{\mathrm{d}\lambda}\qquad S'_{21}(\lambda)\equiv\frac{\mathrm{d}S_{21}(\lambda,\xi)}{\mathrm{d}\lambda}$$

(see § 2). Also, defining b_i, $i=1,2,\ldots,N$, by

$$b_i\equiv S_{21}(\lambda_i,\xi_i) \tag{31}$$

we then have

$$\frac{\delta b_i}{\delta u(x)}=S'_{21}(\lambda_i)\frac{\delta\lambda_i}{\delta u(x)}+\frac{S_{21}(\lambda_i,\xi_i)}{\delta u(x)}. \tag{32}$$

In the following we will use the variational derivatives (28), (29), (30) and (32) to calculate the commutators $\{S_{11}(\lambda,\xi),S_{21}(\lambda',\xi')\}$, $\{S_{11}(\lambda,\xi),S_{11}(\lambda',\xi')\}$, $\{S_{21}(\lambda,\xi),S_{21}(\lambda',\xi')\}$ and $\{\ln\lambda_i,\ln b_j\}$. We first calculate $\{S_{11}(\lambda,\xi),S_{21}(\lambda',\xi')\}$. From (22) we have

$$\{S_{11}(\lambda,\xi),S_{21}(\lambda',\xi')\}=\tfrac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\delta S_{11}(\lambda,\xi)}{\delta u(x)}\frac{\partial}{\partial x}\frac{\delta S_{21}(\lambda',\xi')}{\delta u(x)}\right.$$
$$\left.-\frac{\delta S_{21}(\lambda',\xi')}{\delta u(x)}\frac{\partial}{\partial x}\frac{\delta S_{11}(\lambda,\xi)}{\delta u(x)}\right)\mathrm{d}x. \tag{33}$$

Substituting (28) and (29) into (33) we obtain

$$\{S_{11}(\lambda,\xi),S_{21}(\lambda',\xi')\}$$
$$=\frac{-1}{8\xi(\lambda-\xi)\xi'(\lambda'-\xi')}\int_{-\infty}^{\infty}\left(\phi_1^-(\lambda,\xi,x)^{\mathrm{T}}\right.$$
$$\cdot\phi_2^+(\lambda,\xi,x)\frac{\partial}{\partial x}\phi_1^-(\lambda',\xi',x)\phi_1^+(\lambda',\xi',x)$$
$$-\phi_1^-(\lambda',\xi',x)^{\mathrm{T}}$$
$$\left.\cdot\phi_1^+(\lambda',\xi',x)\frac{\partial}{\partial x}\phi_1^-(\lambda,\xi,x)^{\mathrm{T}}\phi_2^+(\lambda,\xi,x)\right)\mathrm{d}x. \tag{34}$$

Using the AKNS problem (3) we obtain from (34) that

$$\{S_{11}(\lambda,\xi),S_{21}(\lambda',\xi')\}=\frac{1}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\left(\frac{\lambda-\lambda'}{\lambda+\lambda'}\cdot Z_2+\frac{\lambda+\lambda'}{\lambda-\lambda'}Z_1\right)\Bigg|_{x=-\infty}^{x=\infty} \tag{35}$$

where

$$Z_1=\phi_2^+(\lambda,\xi)^{\mathrm{T}}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\phi_1^+(\lambda',\xi')\cdot\phi_1^-(\lambda,\xi)^{\mathrm{T}}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\phi_1^-(\lambda',\xi')$$
$$Z_2=\phi_2^+(\lambda,\xi)^{\mathrm{T}}\cdot\phi_1^+(\lambda',\xi')\cdot\phi_1^-(\lambda,\xi)^{\mathrm{T}}\cdot\phi_1^-(\lambda',\xi'). \tag{36}$$

Finally, applying the boundary conditions (5a) and (27) to (35) we obtain the following commutation relations (see the appendix):

$$\{\ln S_{11}(\lambda,\xi),\ln S_{21}(\lambda',\xi')\} = \left(-b^2\cdot\frac{\lambda^2+\lambda'^2}{\lambda^2-\lambda'^2}-2\cdot\frac{\lambda^2\lambda'^2}{\lambda^2-\lambda'^2}\right)\frac{1}{\xi\xi'}+\pi\mathrm{i}\frac{\lambda^2}{\xi}\delta(\xi-\xi')-\pi\mathrm{i}\frac{\lambda^2}{\xi}\delta(\xi+\xi') \tag{37}$$

where λ and λ' are either real or pure imaginary and

$$\{S_{11}(\lambda_i,\xi_i),S_{21}(\lambda_j,\xi_j)\} = -\lambda_i S_{21}(\lambda_i,\xi_i)S'_{11}(\lambda_i)\cdot\delta_{ij} \tag{38}$$

for $i,j=1,2,\ldots,N$.

Next, we consider the commutator $\{S_{11}(\lambda,\xi),S_{11}(\lambda',\xi')\}$. From (22) we have

$$\{S_{11}(\lambda,\xi),S_{11}(\lambda',\xi')\} = \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\delta S_{11}(\lambda,\xi)}{\delta u(x)}\frac{\partial}{\partial x}\frac{\delta S_{11}(\lambda',\xi')}{\delta u(x)} - \frac{\delta S_{11}(\lambda',\xi')}{\delta u(x)}\frac{\partial}{\partial x}\frac{\delta S_{11}(\lambda,\xi)}{\delta u(x)}\right)\mathrm{d}x. \tag{39}$$

Substituting (28) into (39) we obtain

$$\{S_{11}(\lambda,\xi),S_{11}(\lambda',\xi')\} = \frac{-1}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\int_{-\infty}^{\infty}\Big(\phi_1^-(\lambda,\xi,x)^{\mathrm{T}}\cdot\phi_2^+(\lambda,\xi,x)\frac{\partial}{\partial x}\phi_1^-(\lambda',\xi',x)^{\mathrm{T}}\cdot\phi_2^+(\lambda',\xi',x) - \phi_1^-(\lambda',\xi',x)^{\mathrm{T}}\cdot\phi_2^+(\lambda',\xi',x)\frac{\partial}{\partial x}\phi_1^-(\lambda,\xi,x)^{\mathrm{T}}\cdot\phi_2^+(\lambda,\xi,x)\Big)\,\mathrm{d}x. \tag{40}$$

Using the AKNS problem (3) we obtain from (40) that

$$\{S_{11}(\lambda,\xi),S_{11}(\lambda',\xi')\} = \frac{1}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\left(\frac{\lambda+\lambda'}{\lambda-\lambda'}\tilde{Z}_1+\frac{\lambda-\lambda'}{\lambda+\lambda'}\tilde{Z}_2\right)\Bigg|_{-\infty}^{\infty} \tag{41}$$

where

$$\begin{aligned}\tilde{Z}_1 &= \phi_2^+(\lambda,\xi)^{\mathrm{T}}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\phi_2^+(\lambda',\xi')\cdot\phi_1^-(\lambda,\xi)^{\mathrm{T}}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\phi_1^-(\lambda',\xi')\\ \tilde{Z}_2 &= \phi_2^+(\lambda,\xi)^{\mathrm{T}}\cdot\phi_1^+(\lambda',\xi')\cdot\phi_1^-(\lambda,\xi)^{\mathrm{T}}\cdot\phi_1^-(\lambda',\xi').\end{aligned} \tag{42}$$

Applying the boundary conditions (5a) and (27) to (41) we obtain the following commutation relations (see the appendix):

$$\{S_{11}(\lambda_i,\xi_i),S_{11}(\lambda_j,\xi_j)\}=0 \qquad i,j=1,2,\ldots,N \tag{43}$$

and

$$\{\ln S_{11}(\lambda,\xi),\ln S_{11}(\lambda',\xi')\}=0 \tag{44}$$

where λ and λ' are either real or pure imaginary.

By a similar calculation to those outlined above we can also obtain the following commutation relations:

$$\{S_{21}(\lambda_i,\xi_i),S_{21}(\lambda_j,\xi_j)\}=0 \qquad i,j=1,2,\ldots,N \tag{45}$$

and

$$\{\ln S_{21}(\lambda, \xi), \ln S_{21}(\lambda', \xi')\} = 0 \tag{46}$$

where λ and λ' are either real or pure imaginary.

Using the commutation relations (37), (38) and (43)-(46) we arrive at the following canonical commutation relations:

$$\{Q_\lambda, P_{\lambda'}\} = \delta(\xi - \xi') \qquad \{Q_\lambda, Q'_\lambda\} = 0 \qquad \{P_\lambda, P'_\lambda\} = 0 \tag{47}$$

and

$$\{Q_i, P_j\} = \delta_{ij} \qquad \{Q_i, Q_j\} = 0 \qquad \{P_i, P_j\} = 0 \tag{48}$$

for $i, j = 1, 2, \ldots, N$, where

$$P_\lambda = \frac{-\xi}{\pi\lambda^2} \ln|S_{11}(\lambda, \xi)| \qquad Q_\lambda = \arg S_{21}(\lambda, \xi) \tag{49}$$

where λ is either (i) real positive or (ii) pure imaginary with $0 < -\mathrm{i}\lambda < |b|$, and

$$P_i = -\ln \lambda_i \qquad Q_i = \ln S_{21}(\lambda_i, \xi_i) \qquad i = 1, 2, \ldots, N. \tag{50}$$

Suppose that, among the N eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$, we have that μ_k, $k = 1, \ldots, r$, are pure imaginary and that σ_v, $v = \pm 1, \pm 2, \ldots, \pm S$, such that $\operatorname{Re} \sigma_v \neq 0$ and

$$\sigma_{-v} = -\sigma_v^*. \tag{51}$$

That is, $\{\mu_1, \ldots, \mu_r, \sigma_{\pm 1}, \sigma_{\pm 2}, \ldots, \sigma_{\pm s}\} = \{\lambda_1, \ldots, \lambda_N\}$ and $N = r + 2S$. The variables P_i and Q_i which correspond to the σ_v are not real (i.e. they are complex quantities). For the convenience of later discussion we introduce the following real variables:

$$p_k = -\ln|\mu_k| \qquad q_k = \ln|S_{21}(\mu_k, \xi(\mu_k))| \qquad k = 1, \ldots, r \tag{52}$$

and

$$n_v = -\ln|\sigma_v| \qquad \varphi_v = 2 \ln|S_{21}(\sigma_v, \xi(\sigma_v))| \tag{53}$$

$$\eta_v = -\arg \sigma_v \qquad \psi_v = -2 \arg S_{21}(\sigma_v, \xi(\sigma_v)) \tag{54}$$

$$v = 1, 2, \ldots, S.$$

Using the commutation relations (48) we obtain the following canonical commutation relations:

$$\begin{gathered} \{q_k, p_{k'}\} = \delta_{k,k'} \\ \{q_k, q_{k'}\} = 0 \qquad \{p_k, p_{k'}\} = 0 \end{gathered} \tag{55}$$

for $k, k' = 1, 2, \ldots, r$, and

$$\{\varphi_v, n_{v'}\} = \delta_{v,v'} \qquad \{\varphi_v, \varphi_{v'}\} = 0 \qquad \{n_v, n_{v'}\} = 0 \tag{56}$$

$$\{\psi_v, \eta_{v'}\} = \delta_{v,v'} \qquad \{\psi_{v'}, \psi_{v'}\} = 0 \qquad \{\eta_v, \eta_{v'}\} = 0 \tag{57}$$

for v, $v' = 1, 2, \ldots, S$, and all other commutators are zero.

Now in view of the canonical commutation relations (47) and (55)-(57) we conclude that the inverse scattering transform for the MKdV equation under the non-vanishing condition (2) is a canonical transformation and the scattering data are essentially a set of canonical variables.

Next, we will determine the dependence of the Hamiltonian H (defined by (19)) on the canonical variables P_λ, Q_λ, p_k, q_k, n_v, φ_v, η_v and ψ_v. In view of the boundary condition (27) we have for Im $\xi > 0$ that

$$S_{11}(\lambda, \xi) = \lim_{x\to\infty} \frac{1}{\lambda - \xi} \phi_{12}^-(\lambda, \xi) \exp(\mathrm{i}\xi x) \tag{58}$$

where $\phi_{12}^-(\lambda, \xi)$ is the second component of $\phi_1^-(\lambda, \xi)$. From (58) we get

$$\ln S_{11}(\lambda, \xi) = \int_{-\infty}^{\infty} \sigma_x(x, \lambda)\, \mathrm{d}x \tag{59}$$

where

$$\sigma(x, \lambda) = \ln\left(\frac{1}{\lambda - \xi} \phi_{12}^-(\lambda, \xi) \cdot \exp(\mathrm{i}\xi x)\right). \tag{60}$$

Also, in view of the fact that $\phi_1^-(\lambda, \xi, x)$ satisfies the AKNS problem (3) we arrive at the following Ricatti equation in $\sigma_x - \mathrm{i}\xi - \mathrm{i}\lambda$:

$$u\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sigma_x - \mathrm{i}\xi - \mathrm{i}\lambda}{u}\right) + (\sigma_x - \mathrm{i}\xi - \mathrm{i}\lambda)^2 + u^2 = -2\mathrm{i}\lambda(\sigma_x - \mathrm{i}\xi - \mathrm{i}\lambda). \tag{61}$$

Using the Ricatti equation (61) we can obtain the following power series expansion of $\sigma_x - \mathrm{i}\xi - \mathrm{i}\lambda$:

$$\sigma_x - \mathrm{i}\xi - \mathrm{i}\lambda = f_{-1}(x)\lambda + f_0(x) + \sum_{n=1}^{\infty} \frac{f_n(x)}{(2\mathrm{i}\lambda)^n} \tag{62}$$

where

$$\begin{aligned}
f_{-1} &= -2\mathrm{i} \\
f_0 &= u_x/u \\
f_1 &= u^2 + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_x}{u}\right) \\
f_2 &= -uu_x + \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\left(\frac{u_x}{u}\right)^2 + 2u^2 + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_x}{u}\right)\right] \\
f_3 &= uu_{xx} + u^4 + \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{3}\left(\frac{u_x}{u}\right)^3 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_x}{u}\right)^2\right] + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{1}{2}\left(\frac{u_x}{u}\right)^2 + 2u^2 + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_x}{u}\right)\right].
\end{aligned} \tag{63}$$

From (62), (63) and the power series expansion of $\xi = (\lambda^2 - \lambda_0^2)^{1/2}$:

$$\xi = \lambda + \sum_{n=1}^{\infty} a_n \lambda^{-n}$$

where

$$\begin{aligned}
&a_1 = \tfrac{1}{2}b^2 \qquad a_2 = 0 \qquad a_3 = -\tfrac{1}{8}b^4 \\
&a_4 = 0 \qquad a_5 = \tfrac{1}{16}b^6 \qquad \text{etc}
\end{aligned} \tag{64}$$

we arrive at

$$\ln S_{11}(\lambda, \xi) = \sum_{n=0}^{\infty} \frac{C_{2n+1}}{(2\mathrm{i}\lambda)^{2n+1}} \tag{65}$$

where

$$C_1 = \int_{-\infty}^{\infty} (-b^2 + u^2)\, dx$$
$$C_3 = \int_{-\infty}^{\infty} (-b^4 - u_x^2 + u^4)\, dx \tag{66}$$
$$\dots$$

On the other hand, using (16) and contour integration in the region Im $\xi > 0$ of the upper Riemann sheet, we have the following formula for ln $S_{11}(\lambda, \xi)$:

$$\ln S_{11}(\lambda, \xi) = \sum_{i=1}^{N} \ln\left(\frac{\lambda - \lambda_i}{\lambda - \lambda_i^*}\right) + \frac{2\lambda}{\pi \mathrm{i}} \int_0^{\infty} \frac{\ln|S_{11}(k, (k^2+b^2)^{1/2})|}{k^2 - \lambda^2}\, dk$$
$$- \frac{2\lambda}{\pi \mathrm{i}} \int_0^{|b|} \frac{\arg S_{11}(\mathrm{i}k, (b^2 - k^2)^{1/2})}{k^2 + \lambda^2}\, dk. \tag{67}$$

Expanding (67) in power series of λ^{-1} we obtain

$$\ln S_{11}(\lambda, \xi) = \sum_{n=1}^{\infty} \sum_{i=1}^{N} \frac{1}{n} [(\lambda_i^*)^n - (\lambda_i)^n] \cdot \lambda^{-n}$$
$$- \frac{2}{\pi \mathrm{i}} \sum_{n=0}^{\infty} \frac{1}{\lambda^{2n+1}} \cdot \int_0^{\infty} k^{2n} \ln|S_{11}(k, (k^2+b^2)^{1/2})|\, dk$$
$$- \frac{2}{\pi \mathrm{i}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda^{2n+1}} \cdot \int_0^{|b|} k^{2n} \arg S_{11}(\mathrm{i}k, (b^2 - k^2)^{1/2})\, dk. \tag{68}$$

Remembering that $\{\lambda_1, \dots, \lambda_N\} = \{\mu_1, \dots, \mu_r, \sigma_{\pm 1}, \dots, \sigma_{\pm s}\}$, where $\mu_1, \dots, \mu_r$ are pure imaginary, Re $\sigma_v \neq 0$ and $\sigma_{-v} = -\sigma_v^*$, we have from (68)

$$\ln S_{11}(\lambda, \xi) = \sum_{n=0}^{\infty} \sum_{k=1}^{r} \frac{-2}{2n+1} (\mu_k)^{2n+1} \cdot (\lambda)^{-(2n+1)}$$
$$+ \sum_{n=0}^{\infty} \sum_{v=1}^{S} \frac{2}{2n+1} [(\sigma_v^*)^{2n+1} - (\sigma_v)^{2n+1}] \cdot (\lambda)^{-(2n+1)}$$
$$- \frac{2}{\pi \mathrm{i}} \sum_{n=0}^{\infty} \frac{1}{\lambda^{2n+1}} \cdot \int_0^{\infty} k^{2n} \ln|S_{11}(k, (k^2+b^2)^{1/2})|\, dk$$
$$- \frac{2}{\pi \mathrm{i}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda^{2n+1}} \cdot \int_0^{|b|} k^{2n} \arg S_{11}(\mathrm{i}k, (b^2 - k^2)^{1/2})\, dk. \tag{68a}$$

From (49), (52)-(54) and (68*a*) we arrive at

$$C_{2n+1} = \sum_{k=1}^{r} \frac{(2)^{2n+2}}{2n+1} [\exp(-p_k)]^{2n+1}$$
$$+ \sum_{v=1}^{s} \frac{(2)^{2n+3}}{2n+1} (-1)^{n+1} \exp[-(2n+1)n_v] \cdot \sin(2n+1)\eta_v$$
$$+ (2)^{2n+2} \cdot (-1)^n \cdot \int_0^{\infty} \frac{k^{2n+2}}{(k^2+b^2)^{1/2}} P_k\, dk$$
$$- \frac{1}{\pi} \cdot (2)^{2n+2} \cdot \int_0^{|b|} k^{2n} \arg S_{11}(\mathrm{i}k, (b^2 - k^2)^{1/2})\, dk \tag{69}$$

for $n = 0, 1, 2, \dots$.

Using (19), (66) and (69) we arrive at the following expression for the Hamiltonian H:

$$H = -\tfrac{8}{3}\sum_{k=1}^{r}\exp(-3p_k) - \tfrac{16}{3}\sum_{v=1}^{s}\exp(-3n_v)\sin 3\eta_v + 8\int_0^\infty \frac{k^4}{(k^2+b^2)^{1/2}} P_k\,\mathrm{d}k + \frac{8}{\pi}\cdot\int_0^{|b|} k^2 \arg S_{11}(\mathrm{i}k, (b^2-k^2)^{1/2})\,\mathrm{d}k. \tag{70}$$

We now come to prove that the Hamiltonian H is a function of the momentum variables P_λ, p_k, n_v and η_v only, so that the set of canonical variables P_λ, Q_λ, p_k, q_k, n_v, φ_v, η_v and ψ_v is of action-angle type. We see that in the formula (70) the first three terms are functions of the momentum variables only, so we only need to consider the last term in (70), i.e.

$$\frac{8}{\pi}\int_0^{|b|} k^2 \arg S_{11}(\mathrm{i}k, (b^2-k^2)^{1/2})\,\mathrm{d}k.$$

From (67) we obtain for real positive λ that

$$\arg S_{11}(\lambda,\xi) = -\arg\left(\prod_{i=1}^{N}\frac{\lambda-\lambda_i^*}{\lambda-\lambda_i}\right) + 2\lambda\int_0^\infty \frac{k^2}{(k^2+b^2)^{1/2}}\frac{P_k}{k^2-\lambda^2}\,\mathrm{d}k + \frac{2\lambda}{\pi}\int_0^{|b|}\frac{\arg S_{11}(\mathrm{i}k, (b^2-k^2)^{1/2})}{k^2+\lambda^2}\,\mathrm{d}k \tag{71}$$

and for pure imaginary λ (with $0 < -\mathrm{i}\lambda < |b|$) that

$$\frac{-\pi\lambda^2}{(b^2+\lambda^2)^{1/2}} P_\lambda = 2\mathrm{i}\lambda\int_0^\infty \frac{k^2}{(k^2+b^2)^{1/2}}\frac{P_k}{k^2-\lambda^2}\,\mathrm{d}k - \frac{2\lambda}{\pi\mathrm{i}}\int_0^{|b|}\frac{\arg S_{11}(\mathrm{i}k, (b^2-k^2)^{1/2})}{k^2+\lambda^2}\,\mathrm{d}k. \tag{72}$$

Using (52)-(54) we have for real λ

$$-\arg\left(\prod_{i=1}^{N}\frac{\lambda-\lambda_i^*}{\lambda-\lambda_i}\right) = \sum_{n=0}^{\infty}\frac{1}{\lambda^{2n+1}}\left(\sum_{k=1}^{r}\frac{2(-1)^{n+1}}{2n+1}\exp[-(2n+1)p_k]\right) + \sum_{n=0}^{\infty}\frac{1}{\lambda^{2n+1}}\left(\sum_{v=1}^{s}\frac{4}{2n+1}\exp[-(2n+1)n_v]\sin(2n+1)\eta_v\right). \tag{73}$$

Substituting (73) into (71) we then obtain for real λ that

$$\begin{aligned}\arg S_{11}(\lambda,\xi) = {} & \sum_{n=0}^{\infty}\frac{1}{\lambda^{2n+1}}\left(\sum_{k=1}^{r}\frac{2(-1)^{n+1}}{2n+1}\exp[-(2n+1)p_k]\right) \\ & + \sum_{n=0}^{\infty}\frac{1}{\lambda^{2n+1}}\left(\sum_{v=1}^{s}\frac{4}{2n+1}\exp[-(2n+1)n_v]\sin(2n+1)\eta_v\right) \\ & + 2\lambda\int_0^\infty \frac{k^2}{(k^2+b^2)^{1/2}}\cdot\frac{P_k}{k^2-\lambda^2}\,\mathrm{d}k \\ & + \frac{2\lambda}{\pi}\int_0^{|b|}\frac{\arg S_{11}(\mathrm{i}k, (b^2-k^2)^{1/2})}{k^2+\lambda^2}\,\mathrm{d}k.\end{aligned} \tag{74}$$

We see that (72) and (74) form a system of linear equations in arg $S_{11}(\lambda, \xi)$, where λ is real positive or pure imaginary (with $0<-\mathrm{i}\lambda<|b|$). In principle we can solve the system of linear equations (72) and (74) for arg $S_{11}(\lambda, \xi)$. In view of the fact that the coefficients of the system (72) and (74) are functions of the momentum variables P_λ, p_k, n_v, η_v only, we conclude that arg $S_{11}(\lambda, \xi)$ (where λ is real positive or pure imaginary with $0<-\mathrm{i}\lambda<|b|$) are functions of the momentum variables only. Hence the last term in (70), i.e.

$$\frac{8}{\pi}\int_0^{|b|} k^2 \arg S_{11}(\mathrm{i}k, (b^2-k^2)^{1/2})\,\mathrm{d}k$$

is a function of the momentum variables only.

We then conclude that the Hamiltonian H (given by (70)) is a function of the momentum variables P_λ, p_k, n_v and η_v only. So the set of canonical variables P_λ, Q_λ, p_k, q_k, n_v, φ_v, η_v and ψ_v is of action-angle type. Hence, under the non-vanishing condition (2) the MKdV equation is completely integrable.

5. Conclusions

We have succeeded in showing that the MKdV equation $u_t+6u^2u_x+u_{xxx}=0$ represents a Hamiltonian system under the non-vanishing condition (2), i.e. we rewrote the MKdV equation in the form:

$$u_t=\frac{\partial}{\partial x}\frac{\delta H}{\delta u(x)} \tag{21}$$

where

$$H[u]=-\tfrac{1}{2}\int_{-\infty}^{\infty}[u^4-u_x^2-b^4]\,\mathrm{d}x. \tag{19}$$

The Hamiltonian $H[u]$ is well defined under the non-vanishing condition and it is dependent on the vacuum parameter b. When $b=0$ the Hamiltonian $H[u]$ is identical to that of the vanishing case (Flaschka and Newell 1975, Novikov *et al* 1984).

We have proved that the inverse scattering transform for the MKdV equation under the non-vanishing condition is a canonical transformation. The set of canonical variables, denoted P_λ, Q_λ, p_k, q_k, n_v, φ_v, η_v and ψ_v, are given by (49) and (52)-(54).

We obtained a system of linear equations (given by (72) and (74)) in arg $S_{11}(\lambda, \xi)$, where λ is real positive or pure imaginary (with $0<-\mathrm{i}\lambda<|b|$). The coefficients of the system of linear equations are functions of the momentum variables P_λ, p_k, n_v and η_v only. Hence we conclude that arg $S_{11}(\lambda, \xi)$† are functions of the momentum variables only.

From (49) and (52)-(54) we have

$$|S_{11}(\lambda, \xi)|=\exp\left(-\frac{\pi\lambda^2}{\xi}P_\lambda\right) \tag{75a}$$

$$\arg S_{21}(\lambda, \xi)=Q_\lambda \tag{75b}$$

$$\mu_k=\mathrm{i}\exp(-p_k) \tag{76a}$$

† Where λ is real positive or pure imaginary (with $0<-\mathrm{i}\lambda<|b|$).

$$|S_{21}(\mu_k, \xi(\mu_k))| = \exp(q_k) \tag{76b}$$

$$\sigma_v = \exp[-(n_v + \mathrm{i}\eta_v)] \tag{77a}$$

$$S_{21}(\sigma_v, \xi(\sigma_v)) = \exp[\tfrac{1}{2}(\varphi_v - \mathrm{i}\psi_v)]. \tag{77b}$$

Except for the sign of $S_{21}(\mu_k, \xi(\mu_k))$† we can recover the scattering data from the set of canonical variables P_λ, Q_λ, p_k, q_k, n_v, φ_v, η_v and ψ_v. First of all, $\arg S_{11}(\lambda, \xi)$ is determined since it is a function of the momentum variables. Also, $|S_{11}(\lambda, \xi)|$ is determined using (75a). Now since

$$|S_{11}(\lambda, \xi)|^2 + |S_{21}(\lambda, \xi)|^2 = 1 \tag{18a}$$

for real positive λ and

$$|S_{11}(\lambda, \xi)|^2 - |S_{21}(\lambda, \xi)|^2 = 1 \tag{18b}$$

for pure imaginary λ with $0 < -\mathrm{i}\lambda < |b|$, we can determine $|S_{21}(\lambda, \xi)|$ and hence $S_{21}(\lambda, \xi)$ (since $\arg S_{21}(\lambda, \xi)$ is determined using (75b)). Also, the zeros of $S_{11}(\lambda, \xi)$ in the region $\mathrm{Im}\, \xi > 0$ of the upper Riemann surface, i.e. $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ $(= \{\mu_1, \ldots, \mu_r, \sigma_{\pm 1}, \ldots, \sigma_{\pm s}\})$, are determined using (76a) and (77a). Also, $|S_{21}(\mu_k, \xi(\mu_k))|$ and $S_{21}(\sigma_v, \xi(\sigma_v))$ are determined using (76b) and (77b). Finally, using (67) we can also determine $S'_{11}(\mu_k)$ and $S'_{11}(\sigma_v)$. Hence, except for the sign of $S_{21}(\mu_k, \xi(\mu_k))$, the scattering data are recovered if the canonical variables are known.

By expanding $\ln S_{11}(\lambda, \xi)$ in powers of $1/\lambda$ we obtained an infinite set of conserved integrals $C_{2n+1}[u]$, $n = 0, 1, 2, \ldots,$ of the MKdV equation under the non-vanishing condition (2). We noted that the Hamiltonian $H[u]$ is given by

$$H = -\tfrac{1}{2}C_3. \tag{78}$$

On the other hand, using contour integrations we obtained a formula for $\ln S_{11}(\lambda, \xi)$:

$$\ln S_{11}(\lambda, \xi) = \sum_{i=1}^{N} \ln\left(\frac{\lambda - \lambda_i}{\lambda - \lambda_i^*}\right) + \frac{2\lambda}{\pi \mathrm{i}} \int_0^\infty \frac{\ln|S_{11}(k, (k^2 + b^2)^{1/2})|}{k^2 - \lambda^2}\, \mathrm{d}k$$
$$- \frac{2\lambda}{\pi \mathrm{i}} \int_0^{|b|} \frac{\arg S_{11}(\mathrm{i}k, (b^2 - k^2)^{1/2})}{k^2 + \lambda^2}\, \mathrm{d}k. \tag{67}$$

Then, from this formula we derived an infinite system of linear equations‡ in $\arg S_{11}(\lambda, \xi)$ (where λ is real positive or pure imaginary with $0 < -\mathrm{i}\lambda < |b|$). The coefficients of the system of linear equations are found to be dependent on the momentum variables P_λ, p_k, n_v and η_v only. Hence we conclude that $\arg S_{11}(\lambda, \xi)$ (where λ is real positive or pure imaginary with $0 < -\mathrm{i}\lambda < |b|$) are functions of the momentum variables only. In view of this fact and the two relations $\mu_k = \mathrm{i}\exp(-p_k)$, $\sigma_v = \exp[-(n_v + \mathrm{i}\eta_v)]$ (see (76a) and (77a)) we conclude that the functionals c_{2n+1}, which are the coefficients of the power series expansions of $\ln S_{11}(\lambda, \xi)$ in λ^{-1}, are functions of the momentum variables only. So the Hamiltonian H $(= -\tfrac{1}{2}C_3$, see (78)) is a function of the momentum variables only. Hence the canonical variables P_λ, Q_λ, p_k, q_k, n_v, φ_v, η_v and ψ_v is of action-angle type and the MKdV equation under the non-vanishing condition (2) is completely integrable. We then arrive at the conclusion that the inverse scattering transform for the MKdV equation under the non-vanishing

† In view of the fact that μ_k are pure imaginary, $S_{21}(\mu_k, \xi(\mu_k))$ are real (see (16)).
‡ See (72) and (74).

condition is a canonical transformation, and the scattering data are essentially a set of action-angle variables.

Now consider those equations of the following type:

$$u_t = \frac{\partial}{\partial x}\frac{\delta I}{\delta u(x)} \qquad I[u] = \sum_{n=0}^{M} \alpha_n \cdot C_{2n+1} \tag{79}$$

where α_n are constants and M is a finite positive integer. Note that (79) gives the MKdV equation when $I[u] = -\frac{1}{2}C_3$. In view of the form of (79) we conclude that each of the equations (79) represents a Hamiltonian system, and the corresponding Poisson bracket is the same as that for the MKdV equation, i.e. given by (22). Now, since the functionals C_{2n+1} are functions of the action variables P_λ, p_k, n_v and η_v only, so each of the equations (79) is completely integrable and has the same set of conserved integrals, i.e. $C_{2n+1}[u]$, $n = 0, 1, 2, \ldots$, as the MKdV equation under the non-vanishing condition (2).

Appendix. Derivation of the canonical commutation relations

This appendix gives proofs of the canonical commutation relations discussed in § 4. Applying the boundary conditions (5a) and (27) to (35) we obtain

$$\begin{aligned}
&\{S_{11}(\lambda, \xi), S_{21}(\lambda', \xi')\} \\
&\quad = \frac{(\lambda-\xi)(\lambda'-\xi')+b^2}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot\frac{\lambda+\lambda'}{\lambda-\lambda'} \\
&\quad \times[-\mathrm{i}b(\lambda'-\xi')S_{11}(\lambda,\xi)S_{11}(\lambda',\xi')\exp(-2\mathrm{i}\xi'x) - b^2 S_{11}(\lambda,\xi)\,S_{21}(\lambda',\xi') \\
&\quad +(\lambda-\xi)(\lambda'-\xi')S_{21}(\lambda,\xi)S_{11}(\lambda',\xi')\exp(2\mathrm{i}\xi x-2\mathrm{i}\xi'x) \\
&\quad -\mathrm{i}b(\lambda-\xi)S_{21}(\lambda,\xi)S_{21}(\lambda',\xi')\exp(2\mathrm{i}\xi x) \\
&\quad +\mathrm{i}b(\lambda-\xi)S_{11}(\lambda,\xi)S_{11}(\lambda',\xi')\exp(-2\mathrm{i}\xi'x) \\
&\quad -(\lambda-\xi)(\lambda'-\xi')S_{11}(\lambda,\xi)S_{21}(\lambda',\xi') \\
&\quad +b^2 S_{21}(\lambda,\xi)S_{11}(\lambda',\xi')\exp(2\mathrm{i}\xi x-2\mathrm{i}\xi'x) \\
&\quad +\mathrm{i}b(\lambda'-\xi')S_{21}(\lambda,\xi)S_{21}(\lambda',\xi')\exp(2\mathrm{i}\xi x)]|_{x\to\infty} \\
&\quad -\mathrm{i}b\frac{(\lambda-\xi)+(\lambda'-\xi')}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot\frac{\lambda-\lambda'}{\lambda+\lambda'} \\
&\quad \times[-b^2 S_{11}(\lambda,\xi)S_{11}(\lambda',\xi')\exp(-2\mathrm{i}\xi'x) - \mathrm{i}b(\lambda'-\xi')S_{11}(\lambda,\xi)S_{21}(\lambda',\xi') \\
&\quad -\mathrm{i}b(\lambda-\xi)S_{21}(\lambda,\xi)S_{11}(\lambda',\xi')\exp(2\mathrm{i}\xi x-2\mathrm{i}\xi'x) \\
&\quad +(\lambda-\xi)(\lambda'-\xi')S_{21}(\lambda,\xi)S_{21}(\lambda',\xi')\exp(2\mathrm{i}\xi x) \\
&\quad +(\lambda-\xi)(\lambda'-\xi')S_{11}(\lambda,\xi)S_{11}(\lambda',\xi')\exp(-2\mathrm{i}\xi'x) \\
&\quad -\mathrm{i}b(\lambda-\xi)S_{11}(\lambda,\xi)S_{21}(\lambda',\xi') \\
&\quad -\mathrm{i}b(\lambda'-\xi')S_{21}(\lambda,\xi)S_{11}(\lambda',\xi')\exp(2\mathrm{i}\xi x-2\mathrm{i}\xi'x) \\
&\quad -b^2 S_{21}(\lambda,\xi)S_{21}(\lambda',\xi')\exp(2\mathrm{i}\xi x)]|_{x\to\infty}
\end{aligned}$$

$$
\begin{aligned}
&-\mathrm{i}b\frac{(\lambda-\xi)-(\lambda'-\xi')}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot\frac{\lambda+\lambda'}{\lambda-\lambda'}\\
&\times[-\mathrm{i}b(\lambda'-\xi')S_{21}(-\lambda,-\xi)S_{11}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi x-2\mathrm{i}\xi' x)\\
&+b^2S_{21}(-\lambda,-\xi)S_{21}(\lambda',\xi')\exp(-2\mathrm{i}\xi x)\\
&+(\lambda-\xi)(\lambda'-\xi')S_{11}(\lambda,\xi)S_{11}(\lambda',-\xi')\exp(-2\mathrm{i}\xi' x)\\
&+\mathrm{i}b(\lambda-\xi)S_{11}(\lambda,\xi)S_{21}(\lambda',\xi')\\
&+\mathrm{i}b(\lambda-\xi)S_{21}(-\lambda,-\xi)S_{11}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi x-2\mathrm{i}\xi' x)\\
&+(\lambda-\xi)(\lambda'-\xi')S_{21}(-\lambda,-\xi)S_{21}(\lambda',\xi')\exp(-2\mathrm{i}\xi x)\\
&+b^2S_{11}(\lambda,\xi)S_{11}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi' x)\\
&-\mathrm{i}b(\lambda'-\xi')S_{11}(\lambda,\xi)S_{21}(\lambda',\xi')]|_{x\to-\infty}\\
&-\frac{(\lambda-\xi)(\lambda'-\xi')-b^2}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot\frac{\lambda-\lambda'}{\lambda+\lambda'}\\
&\times[-b^2S_{21}(-\lambda,-\xi)S_{11}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi x-2\mathrm{i}\xi' x)\\
&+\mathrm{i}b(\lambda'-\xi')S_{21}(-\lambda,-\xi)S_{21}(\lambda',\xi')\exp(-2\mathrm{i}\xi x)\\
&-\mathrm{i}b(\lambda-\xi)S_{11}(\lambda,\xi)S_{11}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi' x)\\
&-(\lambda-\xi)(\lambda'-\xi')S_{11}(\lambda,\xi)S_{21}(\lambda',\xi')\\
&+(\lambda-\xi)(\lambda'-\xi')S_{21}(-\lambda,-\xi)S_{11}(-\lambda',\xi')\exp(-2\mathrm{i}\xi x-2\mathrm{i}\xi' x)\\
&+\mathrm{i}b(\lambda-\xi)S_{21}(-\lambda,-\xi)S_{21}(\lambda',\xi')\exp(-2\mathrm{i}\xi x)\\
&-\mathrm{i}b(\lambda'-\xi')S_{11}(\lambda,\xi)S_{11}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi' x)\\
&+b^2S_{11}(\lambda,\xi)S_{21}(\lambda',\xi')]|_{x\to-\infty}.
\end{aligned}
\tag{A1}
$$

From (A1) we arrive at

$$
\{S_{11}(\lambda_i,\xi_i),S_{21}(\lambda_j,\xi_j)\}
=-\lambda_i\cdot S_{21}(\lambda_i,\xi_i)\cdot S'_{11}(\lambda_i)\cdot\delta_{ij}\qquad\text{for}\qquad i,j=1,2,\ldots,N.
\tag{38}
$$

We also obtain from (A1) that

$$
\begin{aligned}
&\{S_{11}(\lambda,\xi),S_{21}(\lambda',\xi')\}\\
&=\frac{1}{\xi\xi'}\left(-b^2\cdot\frac{\lambda^2+\lambda'^2}{\lambda^2-\lambda'^2}-2\cdot\frac{\lambda^2\lambda'^2}{\lambda^2-\lambda'^2}\right)\cdot S_{11}(\lambda,\xi)S_{21}(\lambda',\xi')\\
&+\frac{(\lambda-\xi)(\lambda'-\xi')+b^2}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}(\lambda+\lambda')\\
&\times\left((\lambda-\xi)(\lambda'-\xi')S_{21}(\lambda,\xi)S_{11}(\lambda',\xi')\cdot\frac{\lambda'}{\xi'}\cdot\pi\mathrm{i}\delta(\xi-\xi')\right.\\
&\left.+b^2S_{21}(\lambda,\xi)S_{11}(\lambda',\xi')\cdot\frac{\lambda'}{\xi'}\cdot\pi\mathrm{i}\delta(\xi-\xi')\right)\\
&-\mathrm{i}b\frac{(\lambda-\xi)-(\lambda'-\xi')}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot(\lambda+\lambda')
\end{aligned}
$$

$$\times\left(-\mathrm{i}b(\lambda'-\xi')S_{21}(-\lambda,-\xi)S_{11}(-\lambda',-\xi')\cdot\frac{-\lambda'}{\xi'}\cdot\pi\mathrm{i}\delta(\xi+\xi')\right.$$
$$\left.+\mathrm{i}b(\lambda-\xi)S_{21}(-\lambda,-\xi)S_{11}(-\lambda',-\xi')\cdot\frac{-\lambda'}{\xi'}\cdot\pi\mathrm{i}\delta(\xi+\xi')\right)$$
$$-\frac{(\lambda-\xi)(\lambda'-\xi')-b^2}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot(\lambda-\lambda')$$
$$\times\left(-b^2S_{21}(-\lambda,-\xi)S_{11}(-\lambda',-\xi')\cdot\frac{\lambda'}{\xi'}\cdot\pi\mathrm{i}\delta(\xi+\xi')\right.$$
$$\left.+(\lambda-\xi)(\lambda'-\xi')S_{21}(-\lambda,-\xi)S_{11}(-\lambda',-\xi')\cdot\frac{\lambda'}{\xi'}\cdot\pi\mathrm{i}\delta(\xi+\xi')\right)$$
$$=\left(-b^2\frac{\lambda^2+\lambda'^2}{\lambda^2-\lambda'^2}-2\cdot\frac{\lambda^2\lambda'^2}{\lambda^2-\lambda'^2}\right)\cdot\frac{1}{\xi\xi'}\cdot S_{11}(\lambda,\xi)S_{21}(\lambda',\xi')$$
$$+\frac{\pi\mathrm{i}\lambda^2}{\xi}S_{11}(\lambda,\xi)S_{21}(\lambda',\xi')\delta(\xi-\xi')$$
$$-\frac{\pi\mathrm{i}\lambda^2}{\xi}S_{11}(\lambda,\xi)S_{21}(\lambda',\xi')\delta(\xi+\xi') \qquad \text{(A2)}$$

where λ and λ' are either real or pure imaginary.

From (A2) we immediately have

$$\{\ln S_{11}(\lambda,\xi),\ln S_{21}(\lambda',\xi')\}$$
$$=\frac{1}{\xi\xi'}\left(-b^2\frac{\lambda^2+\lambda'^2}{\lambda^2-\lambda'^2}-2\frac{\lambda^2\lambda'^2}{\lambda^2-\lambda'^2}\right)+\frac{\pi\mathrm{i}\lambda^2}{\xi}\delta(\xi-\xi')-\frac{\pi\mathrm{i}\lambda^2}{\xi}\delta(\xi+\xi'). \qquad (37)$$

Next, we will prove the commutation relations (43) and (44). From ($5a$), (27) and (41) we have

$$\{S_{11}(\lambda,\xi),S_{11}(\lambda',\xi')\}$$
$$=\mathrm{i}b\frac{(\lambda'-\xi')-(\lambda-\xi)}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot\frac{\lambda+\lambda'}{\lambda-\lambda'}$$
$$\times[\mathrm{i}b(\lambda-\xi)S_{11}(\lambda,\xi)S_{11}(\lambda',\xi')$$
$$-(\lambda-\xi)(\lambda'-\xi')S_{11}(\lambda,\xi)S_{21}(\lambda',\xi')\exp(2\mathrm{i}\xi'x)$$
$$+b^2S_{21}(\lambda,\xi)S_{11}(\lambda',\xi')\exp(2\mathrm{i}\xi x)$$
$$+\mathrm{i}b(\lambda'-\xi')S_{21}(\lambda,\xi)S_{21}(\lambda',\xi')\exp(2\mathrm{i}\xi x+2\mathrm{i}\xi'x)$$
$$-\mathrm{i}b(\lambda'-\xi')S_{11}(\lambda,\xi)S_{11}(\lambda',\xi')$$
$$-b^2S_{11}(\lambda,\xi)S_{21}(\lambda',\xi')\exp(2\mathrm{i}\xi'x)$$
$$+(\lambda-\xi)(\lambda'-\xi')S_{21}(\lambda,\xi)S_{11}(\lambda',\xi')\exp(2\mathrm{i}\xi x)$$
$$-\mathrm{i}b(\lambda-\xi)S_{21}(\lambda,\xi)S_{21}(\lambda',\xi')\exp(2\mathrm{i}\xi x+2\mathrm{i}\xi'x)]|_{x\to\infty}$$
$$+\frac{(\lambda-\xi)(\lambda'-\xi')-b^2}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot\frac{\lambda-\lambda'}{\lambda+\lambda'}$$

$$\times[-b^2S_{11}(\lambda,\xi)S_{11}(\lambda',\xi')-\mathrm{i}b(\lambda'-\xi')S_{11}(\lambda,\xi)S_{21}(\lambda',\xi')\exp(2\mathrm{i}\xi'x)$$
$$-\mathrm{i}b(\lambda-\xi)S_{21}(\lambda,\xi)S_{11}(\lambda',\xi')\exp(2\mathrm{i}\xi x)$$
$$+(\lambda-\xi)(\lambda'-\xi')S_{21}(\lambda,\xi)S_{21}(\lambda',\xi')\exp(2\mathrm{i}\xi x+2\mathrm{i}\xi'x)$$
$$+(\lambda-\xi)(\lambda'-\xi')S_{11}(\lambda,\xi)S_{11}(\lambda',\xi')$$
$$-\mathrm{i}b(\lambda-\xi)S_{11}(\lambda,\xi)S_{21}(\lambda',\xi')\exp(2\mathrm{i}\xi'x)$$
$$-\mathrm{i}b(\lambda'-\xi')S_{21}(\lambda,\xi)S_{11}(\lambda',\xi')\exp(2\mathrm{i}\xi x)$$
$$-b^2S_{21}(\lambda,\xi)S_{21}(\lambda',\xi')\exp(2\mathrm{i}\xi x+2\mathrm{i}\xi'x)]|_{x\to\infty}$$
$$-\mathrm{i}b\frac{(\lambda-\xi)-(\lambda'-\xi')}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot\frac{\lambda+\lambda'}{\lambda-\lambda'}$$
$$\times[\mathrm{i}b(\lambda-\xi)S_{21}(-\lambda,-\xi)S_{21}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi x-2\mathrm{i}\xi'x)$$
$$-(\lambda-\xi)(\lambda'-\xi')S_{21}(-\lambda,-\xi)S_{11}(\lambda',\xi')\exp(-2\mathrm{i}\xi x)$$
$$+b^2S_{11}(\lambda,\xi)S_{21}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi'x)$$
$$+\mathrm{i}b(\lambda'-\xi')S_{11}(\lambda,\xi)S_{11}(\lambda',\xi')$$
$$-\mathrm{i}b(\lambda'-\xi')S_{21}(-\lambda,-\xi)S_{21}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi x-2\mathrm{i}\xi'x)$$
$$-b^2S_{21}(-\lambda,-\xi)S_{11}(\lambda',\xi')\exp(-2\mathrm{i}\xi x)$$
$$+(\lambda-\xi)(\lambda'-\xi')S_{11}(\lambda,\xi)S_{21}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi'x)$$
$$-\mathrm{i}b(\lambda-\xi)S_{11}(\lambda,\xi)S_{11}(\lambda',\xi')]|_{x\to-\infty}$$
$$-\frac{(\lambda-\xi)(\lambda'-\xi')-b^2}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot\frac{\lambda-\lambda'}{\lambda+\lambda'}$$
$$\times[-b^2S_{21}(-\lambda,-\xi)S_{21}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi x-2\mathrm{i}\xi'x)$$
$$-\mathrm{i}b(\lambda'-\xi')S_{21}(-\lambda,-\xi)S_{11}(\lambda',\xi')\exp(-2\mathrm{i}\xi x)$$
$$-\mathrm{i}b(\lambda-\xi)S_{11}(\lambda,\xi)S_{21}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi'x)$$
$$+(\lambda-\xi)(\lambda'-\xi')S_{11}(\lambda,\xi)S_{11}(\lambda',\xi')$$
$$+(\lambda-\xi)(\lambda'-\xi')S_{21}(-\lambda,-\xi)S_{21}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi x-2\mathrm{i}\xi'x)$$
$$-\mathrm{i}b(\lambda-\xi)S_{21}(-\lambda,-\xi)S_{11}(\lambda',\xi')\exp(-2\mathrm{i}\xi x)$$
$$-\mathrm{i}b(\lambda'-\xi')S_{11}(\lambda,\xi)S_{21}(-\lambda',-\xi')\exp(-2\mathrm{i}\xi'x)$$
$$-b^2S_{11}(\lambda,\xi)S_{11}(\lambda',\xi')]|_{x\to-\infty}. \tag{A3}$$

From (A3) we arrive at

$$\{S_{11}(\lambda_i,\xi_i),S_{11}(\lambda_j,\xi_j)\}=0 \qquad \text{for} \qquad i,j=1,2,\ldots,N. \tag{43}$$

We also obtain from (A3) that

$$
\begin{aligned}
\{S_{11}(\lambda,\xi), S_{11}(\lambda',\xi')\}
&= ib\frac{(\lambda'-\xi')-(\lambda-\xi)}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot(\lambda+\lambda')\\
&\times\left(ib(\lambda'-\xi')S_{21}(\lambda,\xi)S_{21}(\lambda',\xi')\cdot\frac{-\lambda'}{\xi'}\cdot\pi i\delta(\xi+\xi')\right.\\
&\left.-ib(\lambda-\xi)S_{21}(\lambda,\xi)S_{21}(\lambda',\xi')\cdot\frac{-\lambda'}{\xi'}\cdot\pi i\delta(\xi+\xi')\right)\\
&+\frac{(\lambda-\xi)(\lambda'-\xi')-b^2}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot(\lambda-\lambda')\\
&\times\left((\lambda-\xi)(\lambda'-\xi')S_{21}(\lambda,\xi)S_{21}(\lambda',\xi')\cdot\frac{\lambda'}{\xi'}\cdot\pi i\delta(\xi+\xi')\right.\\
&\left.-b^2S_{21}(\lambda,\xi)S_{21}(\lambda',\xi')\cdot\frac{\lambda'}{\xi'}\cdot\pi i\delta(\xi+\xi')\right)\\
&-ib\frac{(\lambda-\xi)-(\lambda'-\xi')}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot(\lambda+\lambda')\\
&\times\left(ib(\lambda-\xi)S_{21}(-\lambda,-\xi)S_{21}(-\lambda',-\xi')\cdot\frac{-\lambda'}{\xi'}\cdot\pi i\delta(\xi+\xi')\right.\\
&\left.-ib(\lambda'-\xi')S_{21}(-\lambda,-\xi)S_{21}(-\lambda',-\xi')\cdot\frac{-\lambda'}{\xi'}\cdot\pi i\delta(\xi+\xi')\right)\\
&-\frac{(\lambda-\xi)(\lambda'-\xi')-b^2}{8\xi\xi'(\lambda-\xi)(\lambda'-\xi')}\cdot(\lambda-\lambda')\\
&\times\left[-b^2S_{21}(-\lambda,-\xi)S_{21}(-\lambda',-\xi')\cdot\frac{\lambda'}{\xi'}\cdot\pi i\delta(\xi+\xi')\right.\\
&\left.+(\lambda-\xi)(\lambda'-\xi')S_{21}(-\lambda,-\xi)S_{21}(-\lambda',-\xi')\cdot\frac{\lambda'}{\xi'}\cdot\pi i\delta(\xi+\xi')\right].
\end{aligned}
\tag{A4}
$$

Using (14) we obtain from (A4) that

$$\{S_{11}(\lambda,\xi), S_{11}(\lambda',\xi')\}=0 \tag{A5}$$

where λ and λ' are either real or pure imaginary. Equation (A5) implies that

$$\{\ln S_{11}(\lambda,\xi), \ln S_{11}(\lambda',\xi')\}=0. \tag{44}$$

References

Au Yeung T C, Fung P C W and Au C 1984 *J. Phys. A: Math. Gen.* **17** 1425

Flaschka H and Newell A C 1975 *Dynamic Systems: Theory and Applications* ed J Moser (Berlin: Springer) p 355

Gardner C S 1971 *J. Math. Phys.* **12** 1548

Kawata T and Inoue H 1977 *J. Phys. Soc. Japan* **43** 361

—— 1978 *J. Phys. Soc. Japan* **44** 1722

Newell A C 1980 *Solitons* ed R K Bullough and P J Caudry (Berlin: Springer) p 177

Novikov S, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of Solitons, the Inverse Scattering Method* (New York: Plenum)

Wadati M 1972 *J. Phys. Soc. Japan* **32** 1681